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USE OF REFERENCE STATES AND CONSTANT-PROPERTY SOLUTIONS IN PREDICTING MASS-, MOMENTUM-, AND ENERGY-TRANSFER RATES IN HIGH-SPEED LAMINAR FLOWS

ELDON L. KNUTH†

University of California, Los Angeles 24, California

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Abstract—Two simple cases of laminar flow with simultaneous transfers of mass, momentum, and energy are studied first in order to obtain insight into the nature of transport phenomena and thermodynamic conditions existing in high-speed boundary layers involving diffusion of a gas added at the wall. In laminar Couette flow, simplifications are realized as a consequence of the nature of the boundary conditions, and information concerning the effects of differences in the heat capacities of the two species and the effects of deviations of Prandtl number, Schmidt number, and Lewis number from unity are obtained. An exact expression for the reference composition is obtained for the case of isothermal flow; a linearized expression for the reference temperature is obtained for the case in which the viscosity is a linear function of temperature and independent of composition. To first order in blowing rates, the reference composition is the arithmetic average of the compositions at the two surfaces, whereas the expression for the reference enthalpy is a simple extension of Eckert's expression for the reference enthalpy for a laminar boundary-layer with no mass addition at the wall. In laminar boundary-layer flow with constant properties, simplifications are realized as a consequence of the fluid properties being constant properties, and information concerning the effects of the two-dimensional character of boundary-layer flow is obtained. The information obtained for these two simple cases is combined then, and applied to the case of laminar boundary-layer flow with variable fluid properties. Basing all fluid properties upon the reference state suggested in the Couette flow study, mass-, momentum-, and energy-transfer rates and recovery factors for fluids with variable fluid properties, are given as functions of blowing rates for several coolants (including H_2, with molecular weight 2, and I_2, with molecular weight 254) and for several speeds (up to $Ma = 12$) by only two curves to an approximation adequate for many engineering applications. A Reynolds analogy for boundary-layer flows with mass additions at the wall is found to exist.

NOMENCLATURE

B_f, blowing rate for momentum transfer (cf. Table 3);

B_h, blowing rate for energy transfer (cf. Table 3);

B_m, blowing rate for mass transfer (cf. Table 3);

$C_f/2$, $\frac{1}{2}\times$ friction coefficient $\equiv \dfrac{\tau_w}{\rho_\infty u_\infty^2}$;

C_h, Stanton number $\equiv \dfrac{k_w(\partial T/\partial y)_w}{(\rho u c_p)_\infty(T_r - T_w)}$;

C_m, mass-transfer coefficient $\equiv \dfrac{(\rho u)_w(1 - c_w^c)}{(\rho u)_\infty(c_w^c - c_\infty^c)}$;

c^k, mass fraction of component k;

$c^{k\prime}$, dimensionless mass fraction $\equiv \dfrac{c_w^k - c^k}{c_w^k - c^k_\infty}$;

c_p, specific heat at constant pressure;

c_p', dimensionless specific heat $\equiv c_p/c_{p\infty}$;

c_v, specific heat at constant volume;

D, diffusion coefficient in Fick's diffusion law;

† Associate Professor of Engineering.

h, specific enthalpy;
k, thermal conductivity;
k', dimensionless thermal conductivity $\equiv k/k_\infty$;
Le, Lewis number† $\equiv \frac{k}{\rho c_p D}$;
M, molecular weight;
Ma, Mach number;
p, pressure;
Pr, Prandtl number $\equiv \frac{c_p \mu}{k}$;
r, recovery factor $\equiv \frac{2c_{p\infty}(T_r - T_\infty)}{u_\infty^2}$;
R, gas constant;
Re, Reynolds number
$\equiv \rho u_\infty \delta/\mu$ for Couette flow;
$\equiv \rho u_\infty x/\mu$ for boundary-layer flow;
Sc, Schmidt number $\equiv \frac{\mu}{\rho D}$;
T, temperature;
T', dimensionless temperature
$$\equiv \frac{2c_{p\infty}(T - T_w)}{u_\infty^2};$$
T_r, temperature of wall for case in which temperature gradient vanishes at wall;
u, velocity parallel to wall;
u', dimensionless velocity $\equiv u/u_\infty$;
v, velocity normal to wall;
x, co-ordinate parallel to wall;
y, co-ordinate normal to wall;
y', dimensionless co-ordinate $\equiv y/\delta$;
γ, specific-heat ratio $\equiv c_p/c_v$;
δ, thickness of boundary-layer (in case of Couette flow, distance between stationary surface and moving surface);
μ, dynamic viscosity coefficient;
ρ, density;
σ, diameter of molecule;
τ, viscous stress.

† Unfortunately, the definition of the Lewis number is confused in the literature. The definition used here was used, e.g. by Klinkenberg and Mooy [1] in 1948, Bosworth [2] in 1952, Knuth [3] in 1955, Eckert and Drake [4] in 1959, and Rohsenow and Choi [5] in 1961; the inverse definition, i.e. $Le = \rho c_p D/k$, was used, e.g. by Lees [6] in 1956 and Fay and Riddell [7] in 1958.

Superscripts
$'$, dimensionless variable;
$*$, variable evaluated at reference state;
c, mixture components (e.g. coolant) added at wall;
a, mixture components (e.g. air) other than those components added at wall.

Subscripts
w, wall (in case of Couette flow, stationary surface);
∞, free stream (in case of Couette flow, moving surface);
0, limiting value of coefficient as blowing rate approaches zero (cf. Table 2).

INTRODUCTION

CALCULATING exactly the mass-, momentum-, and energy-transfer rates in high-speed boundary-layers is complicated by the dependence of the values of the fluid properties on temperature and composition. Rubesin and Johnson [8], Young and Janssen [9], Eckert [10], and Sommer and Short [11] have noted, however, that the momentum and energy-transfer rates in boundary-layers without mass additions at the wall may be described to good approximation inserting values of properties corresponding to a reference temperature (or enthalpy) in the equations developed for fluids with constant properties (cf. Table 1). Scott [12] attempted to correlate results of several calculations for boundary-layers with mass additions at the wall by supplementing the concept of a reference temperature by the concept of a reference composition. He side-stepped the problem of calculating a reference temperature for the case with mass additions, however, stating that "it seems unlikely that the same correlation equations would apply in the binary boundary-layer case since the temperature profile characteristics are changed drastically" and using, for all flow conditions, property values corresponding to 1450°F. Using a reference coolant concentration of 0·4 the coolant concentration at the wall (the coolant concentration in the free stream being zero), a fair correlation of most calculation results was obtained; unexplained large discrepancies for He–air recovery factors and

Table 1. Several reference–temperature expressions suggested for boundary-layer flows with no mass additions

Conditions	Reference temperature	Ref.	Author
Laminar	$0{\cdot}58\, T_w + 0{\cdot}42\, T_\infty + 0{\cdot}19\, r \dfrac{u_\infty^2}{2c_p^*}$	8	Rubesin and Johnson
Laminar $Ma > 5{\cdot}6$	$0{\cdot}58\, T_w + 0{\cdot}70\, T_\infty + 0{\cdot}14\, r \dfrac{u_\infty^2}{2c_p^*}$	9	Young and Janssen
Turbulent $2{\cdot}5 > Ma > 1{\cdot}5$	$0{\cdot}58\, T_w + 0{\cdot}42\, T_\infty + 0{\cdot}18\, r \dfrac{u_\infty^2}{2c_p^*}$	9	Young and Janssen
Laminar and Turbulent	$0{\cdot}50\, T_w + 0{\cdot}50\, T_\infty + 0{\cdot}22\, r \dfrac{u_\infty^2}{2c_p^*}$	10	Eckert
Turbulent	$0{\cdot}45\, T_w + 0{\cdot}55\, T_\infty + 0{\cdot}20\, r \dfrac{u_\infty^2}{2c_p^*}$	11	Sommer and Short

heat-transfer rates were noted, however. Gross *et al.* [13], examining essentially the same calculation results, presented empirical correlations of the heat-transfer and momentum-transfer rates in which they evaluated the transfer coefficients at free-stream conditions and the blowing parameters at temperatures calculated using the reference–temperature equation given by Eckert [10] for the case with no mass additions. Effects of molecular weight differences were handled empirically by multiplying the blowing parameters by the cube root of the ratio of the air molecular weight to the coolant molecular weight. The concept of reference composition was not used. In the present paper, the physical bases for the concepts of reference temperatures and reference compositions are examined, and methods for calculating reference temperatures and reference compositions for the case with mass additions at the wall are developed.

LAMINAR COUETTE FLOW

Insight into the nature of transport phenomena and thermodynamic conditions existing in high-speed boundary-layers involving diffusion of a gas added at the wall is desired. Although a general description of this system involves cumbersome mathematics obscuring the physical situation, a similar system exists whose description involves only relatively simple mathematics revealing the physical situation. This similar situation is compressible Couette flow, i.e. flow characterized by steady parallel relative motion of two parallel plates containing a viscous compressible fluid (Fig. 1).

Consider the Couette-flow model characterized by the following features:

(1) The velocity of the moving surface, as well as the temperature and concentrations at this surface, are uniform and steady, and are specified.

(2) Heat and mass may pass readily through the moving surface; a steady force, required to maintain steady motion, acts on this surface in the direction of motion.

(3) The momentum flux and the viscous stress in the direction normal to the two surfaces are much smaller than the pressure at some reference plane in the model.

(4) The kinetic energy associated with the mass-weighted average velocity in the direction normal to the two surfaces is much smaller than the enthalpy of the fluid at some reference plane in the model.

(5) Fick's diffusion law describes to good

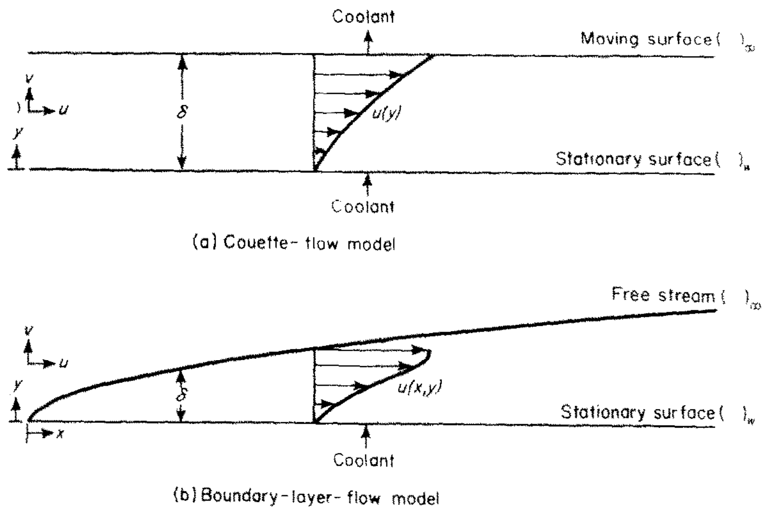

FIG. 1. Comparison of (a) with (b).

approximation the diffusion of the gas added at the wall relative to the rest of the mixture.†

(6) Body force, Dufour, and Soret effects are negligible.

(7) The Prandtl, Schmidt, and Lewis numbers are constants not equal to unity.‡

(8) Viscosity of mixture is constant [it follows from (7) and (8) that the product of density and diffusion coefficient and the ratio of thermal conductivity and specific heat are constants].

(9) Specific heat of gas added at wall and specific heat of rest of mixture are constants (it follows that the specific heat of the mixture is a function of composition only).

This simple model contains the most important features of a high-speed boundary-layer involving mass, momentum, and energy transfers with (a) variable specific heat and thermal conductivity, and (b) arbitrary Prandtl, Schmidt, and Lewis numbers.

Conservations of mass, momentum, and energy for this model, are described by the ordinary differential equations

$$(\rho v)_w = -\rho D \frac{\mathrm{d}c^c}{\mathrm{d}y} + (\rho v)_w c^c$$

$$\tau_w = \mu \frac{\mathrm{d}u}{\mathrm{d}y} - (\rho v)_w u$$

$$k \left.\frac{\mathrm{d}T}{\mathrm{d}y}\right|_w = k \frac{\mathrm{d}T}{\mathrm{d}y} + \mu \frac{\mathrm{d}u^2/2}{\mathrm{d}y} - (\rho v)_w \times \left(h^c + \frac{u^2}{2} - h_w{}^c\right).$$

† Some confusion exists in the literature concerning conditions for which Fick's diffusion law describes concentration diffusion of one species relative to the remaining species in a multi-component mixture. It has been shown [14] that a sufficient condition is that the several binary diffusion coefficients are equal to each other and to the diffusion coefficient in Fick's equation. It is apparent that, for a binary mixture, Fick's law describes concentration diffusion exactly. Note also that, for a multi-component mixture consisting of two groups of species, all species of a group having about the same molecular weight and about the same mutual cross section, Fick's law is not necessarily a good approximation; the restriction on molecular weights is too strong whereas the restriction on collision cross section is too weak. The concentration diffusion of each of these two groups of species, however, is described to good approximation by Fick's equation.

‡ The value of the Prandtl number may be computed, using a model in which the vibrational and rotational energies are transferred by self diffusion, from $Pr = \gamma/(1{\cdot}77\,\gamma - 0{\cdot}45)$. Hirschfelder [15] and Knuth [16] find that this expression describes experimental results from non-polar gases to good approximation. (Note that $0{\cdot}67 \leqslant Pr \leqslant 0{\cdot}76$). The value of the Schmidt number may be computed to good approximation for binary mixtures with small coolant concentrations from

$$Sc = 0{\cdot}76[2M^c/(M^a + M^c)]^{1/2}[(\sigma^a + \sigma^c)/2\sigma^a]^2.$$

The smaller the ratio M^c/M^a is, the smaller is the Schmidt number.

The first equation states that the mass added at the wall is transported across an arbitrary plane by diffusion relative to the mass-weighted average velocity and by convection at the mass-weighted average velocity. The second equation states that the viscous force at the wall equals the viscous force at an arbitrary plane less the force required to accelerate the mass added at the wall to velocity u. The last equation states that the heat conducted into the wall equals the heat crossing an arbitrary plane plus the work done at this plane less the total enthalpy absorbed by the mass added at the wall. In dimensionless form,

$$\frac{\mathrm{d}}{\mathrm{d}y'}\left(1 + B_m \frac{C_{m_0}}{C_m} c^{c'}\right) = B_m\left(1 + B_m \frac{C_{m_0}}{C_m} c^{c'}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}y'}\left(1 + B_f \frac{C_{f_0}}{C_f} u'\right) = B_f\left(1 + B_f \frac{C_{f_0}}{C_f} u'\right)$$

$$k'\frac{\mathrm{d}T'}{\mathrm{d}y'} = k'\frac{\mathrm{d}T'}{\mathrm{d}y'}\bigg|_w - Pr\frac{\mathrm{d}u'^2}{\mathrm{d}y'} + B_h(c_p{}^{c'}T' + u'^2).$$

Integrating the first two equations, one obtains

$$1 + B_m \frac{C_{m_0}}{C_m} c^{c'} = e^{B_m y'}$$

$$1 + B_f \frac{C_{f_0}}{C_f} u' = e^{B_f y'}$$

which, evaluated at $y' = 1$, become

$$1 + B_m \frac{C_{m_0}}{C_m} = e^{B_m}$$

$$1 + B_f \frac{C_{f_0}}{C_f} = e^{B_f}.$$

These four equations may be written also

$$c^{c'} = \frac{e^{B_m y'} - 1}{e^{B_m} - 1}$$

$$u' = \frac{e^{B_f y'} - 1}{e^{B_f} - 1}$$

$$\frac{C_m}{C_{m_0}} = \frac{B_m}{e^{B_m} - 1}$$

$$\frac{C_f}{C_{f_0}} = \frac{B_f}{e^{B_f} - 1}.$$

It is apparent that, for the model being studied, the blowing rate B_m plays the same role in an analysis of the mass transfer as does the blowing rate B_f in an analysis of the momentum transfer.

A suitable integrating factor for the energy-conservation equation is,

$$e^{-B_h y'} c_p{}'^{(B_h/B_m)-1}.$$

Multiplying by this factor, keeping in mind that $k' = c_p{}'$,

$$e^{-B_h y'} c_p{}'^{B_h/B_m} \frac{\mathrm{d}T'}{\mathrm{d}y'} - B_h c_p{}^{c'} e^{-B_h y'} c_p{}'^{(B_h/B_m)-1} T'$$
$$= k'\frac{\mathrm{d}T'}{\mathrm{d}y'}\bigg|_w e^{-B_h y'} c_p{}'^{(B_h/B_m)-1}$$
$$- \left(Pr\frac{\mathrm{d}u'^2}{\mathrm{d}y'} - B_h u'^2\right) {}^{-}e^{B_h y'} c_p{}'^{(B_h/B_m)-1}.$$

Note now that

$$c_p{}' = c_p{}^{c'} + (c_p{}^{a'} - c_p{}^{c'})(1 - c^c)$$
$$= c_p{}^{c'} + (c_p{}^{a'} - c_p{}^{c'})(1 - c_w{}^c)\, e^{B_m y'}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}y'} e^{-B_h y'} c_p{}'^{B_h/B_m} = \frac{\mathrm{d}}{\mathrm{d}y'}(e^{-B_m y'} c_p{}')^{B_h/B_m}$$
$$= -B_h \frac{c_p{}^{c'}}{c_p{}'}(e^{-B_m y'} c_p{}')^{B_h/B_m}$$
$$= -B_h c_p{}^{c'} e^{-B_h y'} c_p{}'^{(B_h/B_m)-1}.$$

Note also that

$$Pr\frac{\mathrm{d}u'^2}{\mathrm{d}y'} - B_h u'^2 = \left(\frac{C_f}{C_{f_0}}\right)^2 \frac{B_h}{B_f^2}(e^{2B_f y'} - 1).$$

Hence the energy-conservation equation may be written

$$\mathrm{d}(e^{-B_h y'} c_p{}'^{B_h/B_m} T')$$
$$= -k'\frac{\mathrm{d}T'}{\mathrm{d}y'}\bigg|_w \frac{1}{B_h c_p{}^{c'}}\,\mathrm{d}(e^{-B_h y'} c_p{}'^{B_h/B_m})$$
$$- \left(\frac{C_f}{C_{f_0}}\right)^2 \frac{B_h}{B_f^2} e^{(2B_f - B_h)y'} c_p{}'^{(B_h/B_m)-1}\,\mathrm{d}y'$$
$$- \left(\frac{C_f}{C_{f_0}}\right)^2 \frac{1}{B_f^2 c_p{}^{c'}}\,\mathrm{d}(e^{-B_h y'} c_p{}'^{B_h/B_m}).$$

Integrating from the stationary plate to the moving plate, one obtains finally,

$$e^{-B_h} T_\infty{}' = -k'\frac{\mathrm{d}T'}{\mathrm{d}y'}\bigg|_w \frac{e^{-B_h} - c_{p_w}{}'^{B_h/B_m}}{B_h c_p{}^{c'}} - e^{-B_h} r$$

which may be arranged into the form

$$\frac{C_h}{C_{h0}} = \frac{B_h \dfrac{c_p{}^c}{c_{p\infty}}}{\left(\dfrac{c_{p_w}}{c_{p\infty}} e^{B_m}\right)^{B_h/B_m} - 1}$$

with the recovery factor r given by

$$r \equiv \left(\frac{C_f}{C_{f_0}}\frac{1}{B_f}\right)^2 \times \left[B_h e^{B_h} \int_0^1 e^{(2B_f - B_h)y'} \left(\frac{c_p}{c_{p\infty}}\right)^{(B_h/B_m)-1} \mathrm{d}y' - \frac{\left(\dfrac{c_{p_w}}{c_{p\infty}} e^{B_m}\right)^{B_h/B_m} - 1}{\dfrac{c_p{}^c}{c_{p\infty}}} \right].$$

It is seen that the exact expression for the Stanton number does not have the form of the expressions for the mass-transfer coefficient and the friction coefficient. However, noting that

$$\frac{c_{p_w}}{c_{p\infty}} e^{B_m} = 1 + \frac{c_p{}^c}{c_{p\infty}} (e^{B_m} - 1)$$

expanding in infinite series, and retaining only linear terms in blowing rates, one may write†

$$\frac{C_h{}^*}{C_{h_0}} \approx 1 - \tfrac{1}{2} B_h \frac{c_p{}^c}{c_p{}^*}$$

with

$$c_p{}^* \approx \frac{c_{p_w} + c_{p\infty}}{2}$$

i.e. to linear approximation, the product $B_h\, c_p{}^c/c_p{}^*$ plays the same role in an analysis of the heat-transfer rate as do the blowing rates B_f and B_m in analyses of the momentum- and mass-transfer rates.

A closed-form expression for the integral appearing in the equation for the recovery factor is not apparent to the author. Consequently, for small blowing rates, the several exponentials are expanded in infinite series to obtain, after considerable mathematical manipulation,

$$\frac{r^*}{r_0} \approx \frac{c_p{}^*}{c_{p\infty}} \left(\frac{C_f}{C_{f_0}}^2\right) \left\{1 + \tfrac{1}{3}\left[(2B_f + B_h) - (B_m - B_h)\frac{c_p{}^c - c_p{}^*}{c_p{}^*}\right]\right\}$$

$$\approx \frac{c_p{}^*}{c_{p\infty}} \left\{1 + \tfrac{1}{3}\left[(B_h - B_f) + (B_h - B_m)\frac{c_p{}^c - c_p{}^*}{c_p{}^*}\right]\right\}$$

$$\approx 1 + \tfrac{1}{3}\left[(B_h - B_f) + (B_h + \tfrac{1}{2} B_m)\frac{c_p{}^c - c_p{}^*}{c_p{}^*}\right]$$

where only terms up to and including linear terms in blowing rates are retained. Note that the recovery factor for the case with mass transfer can be greater than the recovery factor for the case with no mass transfer.

Note that if either the Lewis number is unity ($B_h = B_m$) or the specific heats of both species are equal ($c_p{}^c = c_p{}^a$), then

$$\frac{C_h}{C_{h_0}} = \frac{B_h}{e^{B_h} - 1}$$

$$r = \left(\frac{1}{B_f}\frac{C_f}{C_{f_0}}\right)^2 \frac{B_h(e^{2B_f} - 1) - 2B_f(e^{B_h} - 1)}{2B_f - B_h}$$

whereas, if the Prandtl and Schmidt numbers are unity, then

$$\frac{C_f}{2} = C_m = C_h$$

$$r = 1$$

i.e. the Reynolds analogy does hold for Couette flows with simultaneous transfers of mass, momentum, and energy provided that the several coefficients are defined properly.

Reference values of the fluid properties are desired which give correct values of transport at the stationary surface for flow with variable fluid properties when used in equations for flow with constant fluid properties. In order to obtain

† Use of the reference heat capacity $c_p{}^*$ at this point anticipates the linearized expression for reference composition derived at a subsequent point.

some insight into the manner in which the reference composition is to be computed, consider the equation describing transfer of the species not added at the wall:

$$(\rho v)_w \, c^a - \rho D \frac{dc^a}{dy} = 0.$$

Separating variables and integrating,

$$(\rho v)_w \delta = \int_{c^a_w}^{c^a_\infty} \rho D \frac{dc^a}{c^a}.$$

Consider now the case of isothermal flow. Then, to good approximation,

$$\rho D = \frac{pD}{RT} M$$

with pD/RT independent of composition. The molecular weight may be written as a function of composition by

$$M = \frac{M^a M^c}{M^a + (M^c - M^a)c^a}.$$

Substituting into the integral and carrying out the indicated integration, one obtains

$$(\rho v)_w \, \delta = \frac{pD}{RT} M^c \ln \frac{c^a_\infty}{c^a_w} \frac{M_\infty}{M_w}.$$

Alternatively, if one sets the molecular weight equal to a constant, M^*, and integrates, then

$$(\rho v)_w \, \delta = \frac{pD}{RT} M^* \ln \frac{c^a_\infty}{c^a_w}.$$

Equating the right-hand sides of these two equations,

$$\frac{M^*}{M^c} = 1 + \frac{\ln M_\infty / M_w}{\ln c^a_\infty / c^a_w}.$$

Hence, the reference composition is given by

$$c^{a*} = \frac{1 - M^c/M^*}{1 - M^c/M^a} = \frac{M^a}{M^a - M^c} \frac{\ln M_\infty/M_w}{\ln c^a_\infty M_\infty / c^a_w M_w}$$

which, to first order in coolant mass fractions, may be written

$$c^{c*} \approx \frac{c^c_w + c^c_\infty}{2}$$

i.e. the reference composition is given, to first order, by the average of the compositions at the two surfaces.

In order to obtain some insight into the manner in which the reference temperature is to be computed, consider the equation describing momentum transfer

$$\tau = \mu \frac{du}{dy}.$$

Separating variables and integrating,

$$\delta = \frac{u_\infty}{\tau_w} \int_0^1 \mu \frac{\tau_w}{\tau} \, du'.$$

If one sets the viscosity equal to a constant, μ^*, and integrates, then

$$\delta = \frac{u_\infty \mu^*}{\tau_w} \int_0^1 \frac{\tau_w}{\tau} \, du'.$$

Equating the right-hand sides of these two equations,

$$\mu^* = \frac{\int_0^1 \mu \frac{\tau_w}{\tau} \, du'}{\int_0^1 \frac{\tau_w}{\tau} \, du'}.$$

Since the viscosity is frequently more sensitive to temperature changes than to composition changes, and since the purpose of the immediate calculations is to obtain insight into the concept of a reference temperature, consider the case in which the viscosity is a linear function of temperature and independent of composition. Then

$$\mu^* = \mu(T^*)$$

with

$$T^* = \frac{\int_0^1 T \frac{\tau_w}{\tau} \, du'}{\int_0^1 \frac{\tau_w}{\tau} \, du'}.$$

An expression for the temperature distribution is obtained by integrating the energy equation over the range from the stationary plate to some surface intermediate to the two plates, the result being

$$e^{-B_h y'} c_p'^{\,B_h/B_m} T' = -k' \left.\frac{\mathrm{d}T'}{\mathrm{d}y'}\right|_w \frac{e^{-B_h y'} c_p'^{\,B_h/B_m} - c_{p_w}'^{\,B_h/B_m}}{B_h c_p^{c'}} - \left(\frac{C_f}{C_{f_0}}\frac{1}{B_f}\right)^2 \left[B_h \int_0^{y'} e^{(2B_f-B_h)y'} c_p'^{\,(B_h/B_m)-1}\,\mathrm{d}y' + \frac{e^{-B_h y'} c_p'^{\,B_h/B_m} - c_{p_w}'^{\,B_h/B_m}}{c_p^{c'}}\right]$$

or

$$T' = -k' \left.\frac{\mathrm{d}T'}{\mathrm{d}y'}\right|_w \frac{1 - e^{B_h y'}\left(\frac{c_{p_w}'}{c_p'}\right)^{B_h/B_m}}{B_h c_p^{c'}} - \left(\frac{C_f}{C_{h_0}}\frac{1}{B_f}\right)^2 B_h \left[e^{B_h y'} c_p'^{\,-(B_h/B_m)} \int_0^{y'} e^{(2B_f-B_h)y'} c_p'^{\,(B_h/B_m)-1}\,\mathrm{d}y' + \frac{1 - e^{B_h y'}\left(\frac{c_{p_w}'}{c_p'}\right)^{B_h/B_m}}{B_h c_p^{c'}}\right].$$

Replacing the co-ordinate y' by a function of the velocity u'

$$T' = -k' \left.\frac{\mathrm{d}T'}{\mathrm{d}y'}\right|_w \frac{1 - \left(1 + B_f \frac{C_{f_0}}{C_f} u'\right)^{Pr} \left(\frac{c_{p_w}'}{c_p'}\right)^{B_h/B_m}}{B_h c_p^{c'}} - \left(\frac{C_f}{C_{f_0}}\frac{1}{B_f}\right)^2 B_h \left[\frac{C_{f_0}}{C_f}\left(1 + B_f \frac{C_{f_0}}{C_f} u'\right)^{Pr} c_p'^{\,B_h/B_m} \int_0^{u'} \left(1 + B_f \frac{C_{f_0}}{C_f} u'\right)^{1-Pr} c_p'^{\,(B_h/B_m)-1}\,\mathrm{d}u' + \frac{1 - \left(1 + B_f \frac{C_{f_0}}{C_f} u'\right)^{Pr}\left(\frac{c_{p_w}'}{c_p'}\right)^{B_h/B_m}}{B_h c_p^{c'}}\right].$$

Note that the specific heat also may be written as a function of the velocity u', i.e.

$$c_p = c_p^* + (c_p - c_p^*) = c_p^* + (c_p^c - c_p^*)\left[1 - \frac{2\,e^{B_m y'}}{1 + e^{B_m}}\right] = c_p^* + (c_p^c - c_p^*)\left[1 - 2\frac{\left(1 + B_f \frac{C_{f_0}}{C_f} u'\right)^{Sc}}{(1 + e^{B_m})}\right].$$

Hence, substituting into the expression for temperature, expanding in series and retaining only first-order terms in blowing rates, and integrating, one obtains, after considerable mathematical manipulation,

$$T^* \approx \tfrac{1}{2}(T_w + T_\infty) + \left(\tfrac{1}{12} B_h \frac{c_p^c}{c_p^*} + \tfrac{1}{12} B_m \frac{c_p^c - c_p^*}{c_p^*}\right)(T_w - T_\infty) + \left(\tfrac{1}{2} - \tfrac{1}{12} B_h \frac{c_p^c}{c_p^*} - \tfrac{1}{12} B_m \frac{c_p^c - c_p^*}{c_p^*}\right)(T_r - T_\infty) - \left[\tfrac{1}{3} - \tfrac{1}{6} B_f + \tfrac{1}{12} B_h \frac{c_p^c}{c_p^*}\right] \frac{r_0 u_\infty^2}{2c_p^*}.$$

But,

$$T_r - T_\infty = \frac{r^* u_\infty^2}{2c_p^*} = \frac{r^*}{r_0}\frac{r_0 u_\infty^2}{2c_p^*} \approx \left\{1 + \tfrac{1}{3}\left[(B_h - B_f) + (B_h + \tfrac{1}{2} B_m)\frac{c_p^c - c_p^*}{c_p^*}\right]\right\} \frac{r_0 u_\infty^2}{2c_p^*}.$$

Hence, substituting into the expression for reference temperature,

$$T^* \approx \tfrac{1}{2}(T_w + T_\infty) + \tfrac{1}{6} r_0 \frac{u_\infty^2}{2c_p^*} + \tfrac{1}{12}\left(B_h \frac{c_p^c}{c_p^*} + B_m \frac{c_p^c - c_p^*}{c_p^*}\right)(T_w - T_\infty).$$

Since

$$\frac{c_{p_w}}{c_p^*} \approx 1 + \tfrac{1}{2} B_m \frac{c_p^c - c_p^*}{c_p^*}$$

$$\frac{c_{p_\infty}}{c_p^*} \approx 1 - \tfrac{1}{2} B_m \frac{c_p^c - c_p^*}{c_p^*}$$

one may write alternatively

$$h^* \approx \tfrac{1}{2}(h_w + h_\infty) + \tfrac{1}{6}\, r_0 \frac{u_\infty^2}{2} + \tfrac{1}{12}\left(B_h \frac{c_p^c}{c_p^*} - 2B_m \frac{c_p^c - c_p^*}{c_p^*}\right)(h_w - h_\infty).$$

For zero blowing rate,

$$T^*(B = 0) = \tfrac{1}{2}(T_w + T_\infty) + \tfrac{1}{6}\, r_0 \frac{u_\infty^2}{2c_p^*}$$

which compares favorably with the empirical reference–temperature equations used for laminar two-dimensional boundary layers (cf. Table 1).

LAMINAR BOUNDARY-LAYER FLOW WITH CONSTANT PROPERTIES

Recall that the purpose of the study described in this paper is to develop methods for calculating reference states for flows with variable fluid properties such that engineering calculations for flows with variable fluid properties may be made using these reference states and relations for flows with constant fluid properties. Hence, relations for constant-property laminar boundary-layer flows with mass transfer are required. Furthermore, forms of these relations which are convenient for use by engineers are desirable. Consequently, relations for constant-property flows are discussed briefly here.

A series of curves, indicating, for constant-property flows, the dependence of recovery factors, Stanton numbers, friction coefficients, and wall concentrations on blowing rates, Prandtl numbers, and Schmidt numbers, are presented by Scott [12], Figs. 4 and 5. If the number of curves found in these figures could be reduced to two (one curve of recovery factor vs. blowing rates and one curve of transport rates vs. blowing rates), then use of these results would be facilitated greatly.

An examination of the equations describing Couette flows reveals that, for Couette flows with constant fluid properties, one curve suffices to describe variations of transport rates with blowing rates provided that one plots C_f/C_{f_0} vs. B_f, C_m/C_{m_0} vs. B_m, and C_h/C_{h_0} vs. B_h; retaining only terms up to and including linear terms in blowing rates, one curve suffices to describe variations of recovery factors with blowing rates provided that one plots r/r_0 vs. $B_h - B_f$. Plots of transport rates and recovery factors vs. blowing rates given by Scott are reproduced here as Figs. 2–4. The co-ordinates used in these figures are obtained from parameters appearing in the analysis of Couette flow by making the

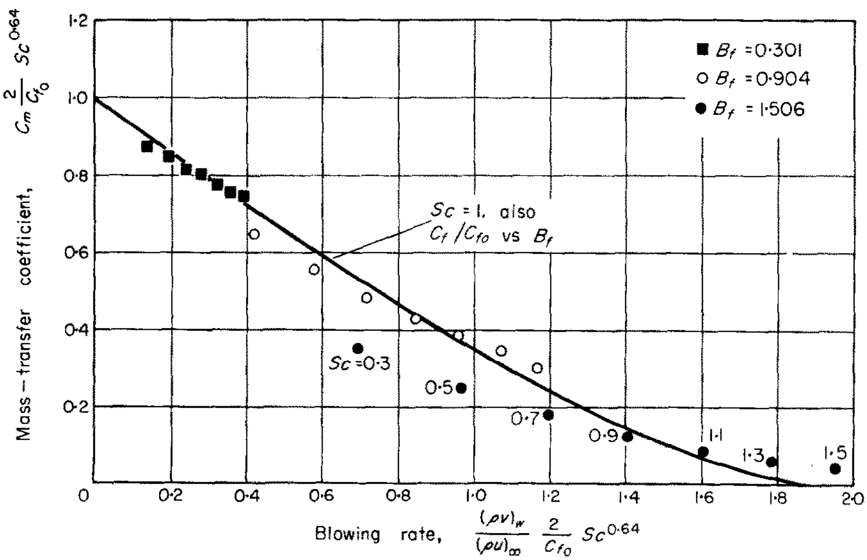

FIG. 2. Friction and mass-transfer coefficients for fluids with constant properties as functions of blowing rates and for several values of Schmidt number.

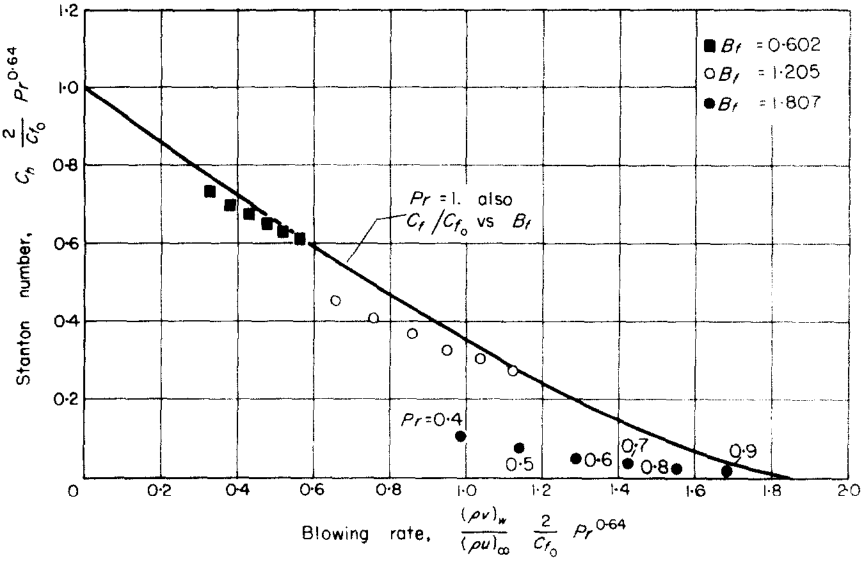

FIG. 3. Friction coefficient and Stanton number for fluids with constant properties as functions of blowing rates and for several values of Prandtl number.

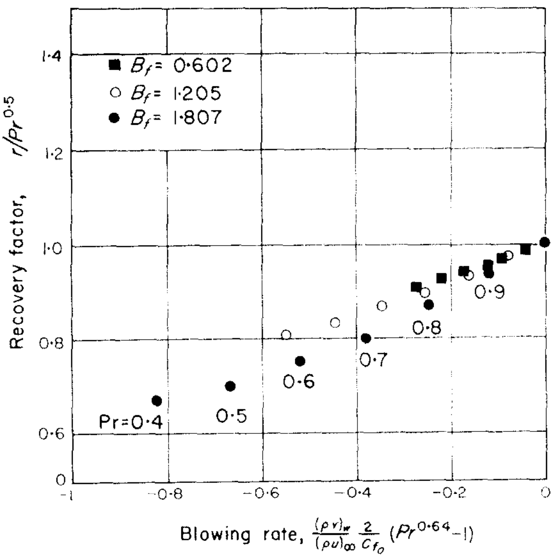

FIG. 4. Recovery factor for fluids with constant properties as function of blowing rate and for several values of Prandtl number.

substitutions for transfer coefficients indicated in Table 2. It is seen that, if either blowing rates are small or Prandtl and Schmidt numbers are near unity, then the several relations are described adequately by two curves (one curve for transport rates and one curve for recovery factors).

The fact that the several relations are not described adequately by only two curves if the blowing rates are large and the Prandtl and

Table 2. Limits of coefficients as blowing rates vanish for fluids with constant properties

	Couette flow	Boundary-layer flow
Friction coefficient, $C_{f0}/2$	$\frac{1}{Re}$	$\frac{0{\cdot}332}{\sqrt{Re}}$
Mass-transfer coefficient, C_{m0}	$\frac{1}{Re}\frac{1}{Sc}$	$\frac{0{\cdot}332}{\sqrt{(Re)}}\frac{1}{Sc^{0{\cdot}64}}$
Stanton number, C_{h0}	$\frac{1}{Re}\frac{1}{Pr}$	$\frac{0{\cdot}332}{\sqrt{(Re)}}\frac{1}{Pr^{0{\cdot}64}}$
Recover factor, r_0	Pr	$\sqrt{Pr}$

Note: For Prandtl and Schmidt numbers near unity, the exponent 0·64 appearing in the limits of the mass-transfer coefficient and Stanton number describes better the results of exact calculations than does the exponent 2/3. The exponent 2/3 was suggested first apparently by Pohlhausen in 1921 [29] to describe results for Prandtl numbers from 0·6 to 15; his results for Prandtl numbers from 0·6 to 1·1 are described better using the exponent 0·64. For a Schmidt number of 0·2 (realized, e.g. for some He–air and H_2–air mixtures), $Sc^{2/3}$ differs from $Sc^{0{\cdot}64}$ by 4 per cent.

Schmidt numbers deviate appreciably from unity is an inherent difference between Couette flows and two-dimensional boundary-layer flows; for Couette flows, the several transport rates (C_f/C_{f_0}, C_m/C_{m_0}, and C_h/C_{h_0}) approach zero only as the respective blowing rates (B_f, B_m, and B_h) approach infinity whereas, for two-dimensional boundary-layer flows, the several transport rates all approach zero as the blowing rate B_f approaches 1·865. In order to describe adequately the several relations for boundary-layer flows using only two curves, the variables suggested by the analysis of Couette flows must be modified. These modifications can be made by replacing the parameters C_{m_0}, C_{h_0}, and r_0 by parameters which vary with the blowing rate B_f. The new parameters must (a) approach C_{m_0}, C_{h_0}, and $Pr^{0\cdot 5}$ as B_f approaches zero, (b) approach $C_{f_0}/2$, $C_{f_0}/2$ and unity as the Prandtl and Schmidt numbers approach unity, and (c) approach $C_{f_0}/2$, $C_{f_0}/2$, and $Pr^{1\cdot 2}$ as B_f approaches 1·865 (the limiting expressions for the parameter replacing r_0 result from an examination of Fig. 4 of [12]). The simplest combination satisfying these conditions is

$$(C_{f_0}/2)[C_{m_0}/(C_{f_0}/2)]^{(C_f/C_{f_0})^m},$$
$$(C_{f_0}/2)[C_{h_0}/(C_{f_0}/2)]^{(C_f/C_{f_0})^m},$$

and $Pr^{1\cdot 2-0\cdot 7(C_f/C_{f_0})^m}$. Results for $m = 1/6$ are presented as Figs. 5–7. In these figures, the several relations are described, for all blowing rates and for Prandtl and Schmidt numbers from 0·5 to 1·5, by only two curves to an approximation adequate for many engineering applications.

LAMINAR BOUNDARY-LAYER FLOW WITH VARIABLE PROPERTIES

In the preceding parts of this paper, two simple cases of laminar flow with simultaneous transfers of mass, momentum and energy are discussed. In the first case (laminar Couette flow), simplifications are realized as a consequence of the nature of the boundary conditions, and information concerning the effects of differences in the heat capacities of the two species and the effects of deviations of Prandtl number, Schmidt number, and Lewis number from unity are obtained. In the second case (laminar boundary-layer flow with constant properties), simplifications are realized as a consequence of the fluid properties being constant properties, and information concerning the effects of the two-dimensional character of boundary-layer flow are obtained. In this part, the information obtained in the preceding parts is combined, and applied to the case of laminar boundary-layer flow with

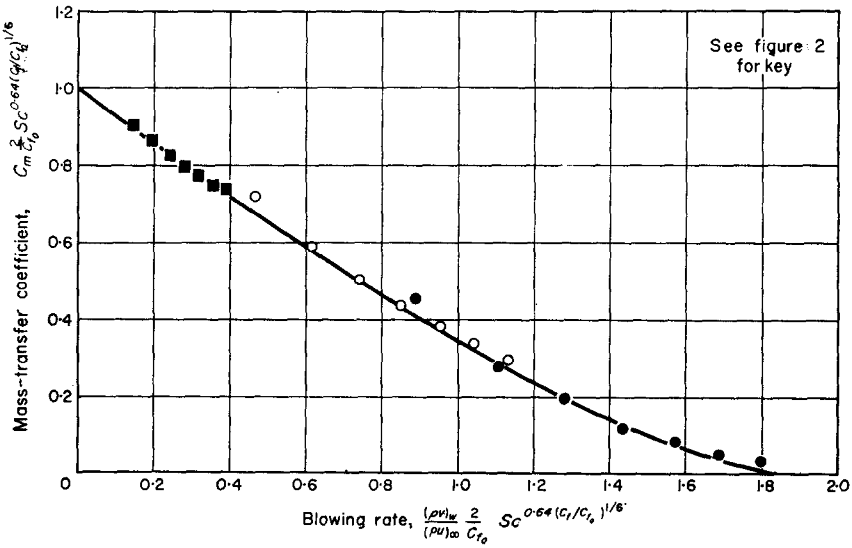

FIG. 5. Friction and mass-transfer coefficients for fluids with constant properties as functions of blowing rates and for several values of Schmidt number: result of attempt to describe results for several Schmidt numbers by a single curve.

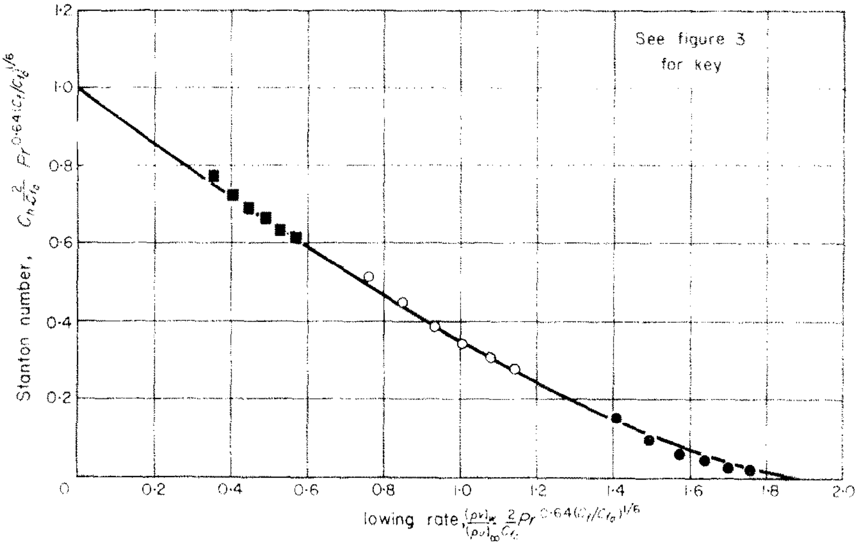

FIG. 6. Friction coefficient and Stanton number for fluids with constant properties as functions of blowing rates and for several values of Prandtl number: result of attempt to describe results for several Prandtl numbers by a single curve.

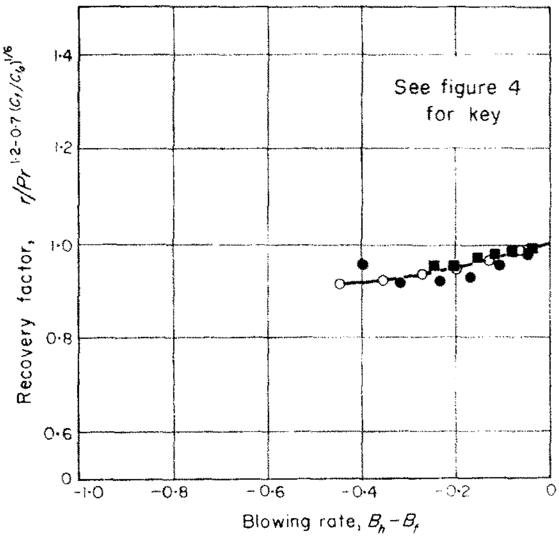

FIG. 7. Recovery factor for fluids with constant properties as function of blowing rate and for several values of Prandtl number: result of attempt to describe results for several Prandtl numbers by a single curve.

variable fluid properties, in an effort to provide a simplified correlation of results for this flow to an approximation adequate for many engineering applications.

Experimental mass-transfer data sufficiently extensive for use in the proposed correlation is difficult to obtain; results of several exact calculations of binary laminar boundary-layer flows are available in the literature. Furthermore, it is believed that for laminar flows, a correlation of results of exact calculations would be just as meaningful as a correlation of experimental results. Hence, only results of available exact calculations are examined here.

Results of exact calculations by Baron [17] for He–air and CO_2–air, by Eckert, Schneider, Hayday, and Larson [18] for H_2–air, by Sziklas [19] for H_2–air, He–air, and H_2O–air, and by Gross [13 and 20] for I_2–air are presented as Figs. 8–11. Co-ordinates are the same as used in Figs. 2 and 3 for fluids with constant properties; fluid properties are based on free-stream conditions. Large deviations from the curves for constant fluid properties are noted.

The parameters used eventually in the description of these boundary-layer flows are compared in Table 3 with the corresponding parameters for Couette flow. The expressions for reference enthalpy h^*, reference temperature T^*, and reference concentration $c^{\alpha *}$ developed for Couette flow, are used essentially unchanged

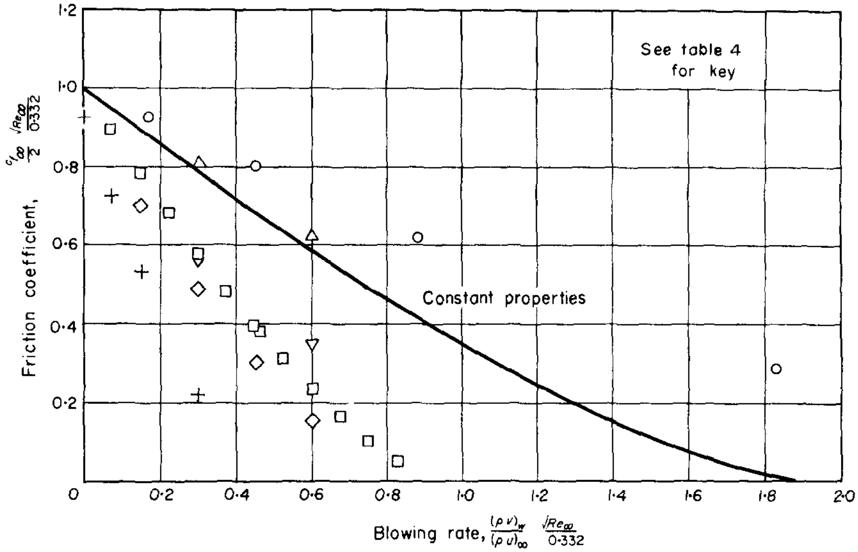

FIG. 8. Friction coefficient for fluids with variable properties as function of blowing rate and for several coolants.

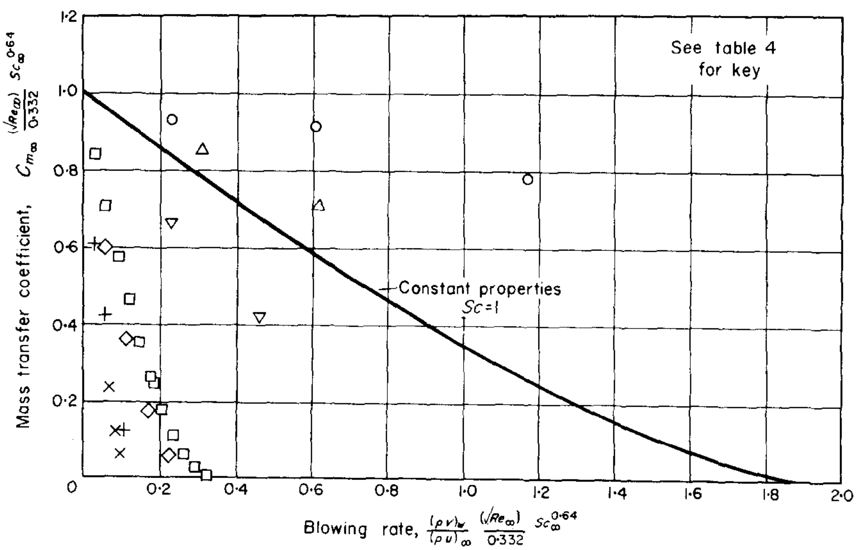

FIG. 9. Mass-transfer coefficient for fluids with variable properties as function of blowing rate and for several coolants.

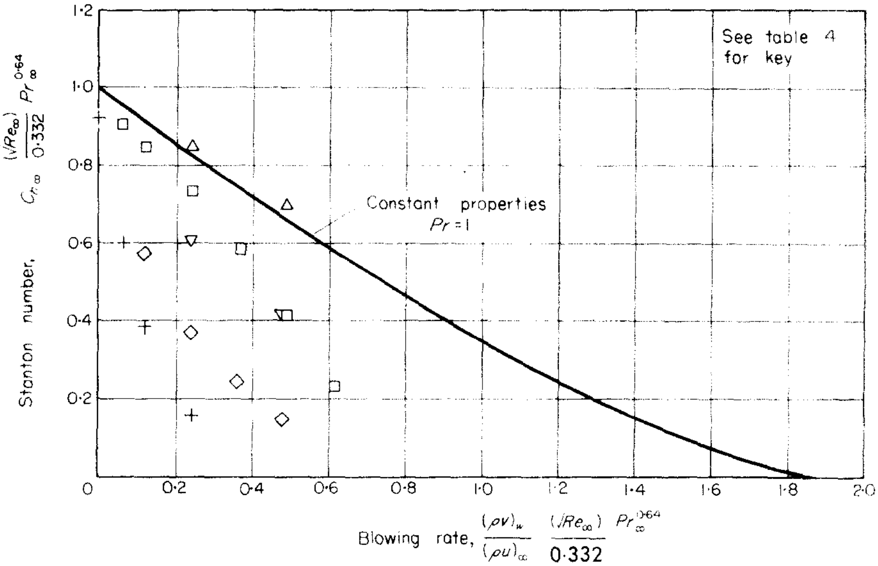

FIG. 10. Stanton number for fluids with variable properties as function of blowing rate and for several coolants.

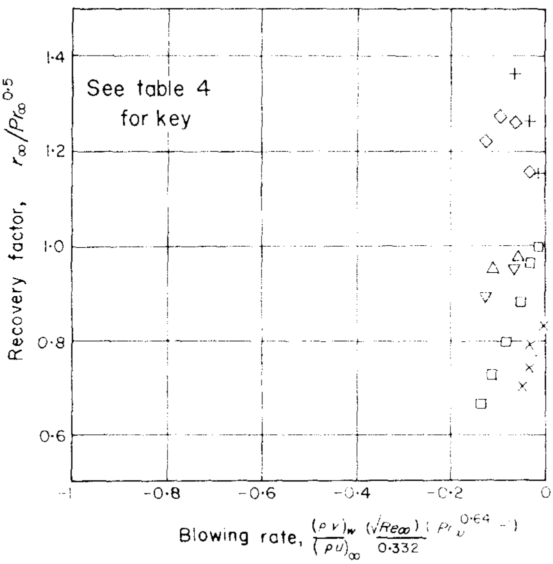

FIG. 11. Recovery factor for fluids with variable properties as function of blowing rate and for several coolants.

for boundary-layer flow; the fractions appearing in the expressions for reference enthalpy and reference temperature are written as decimals and only the first significant figure of each decimal is retained. The reference temperatures are computed using

$$T^* \approx 0{\cdot}5\,(T_w + T_\infty) + 0{\cdot}2\,r_0^* \frac{u_\infty^2}{2c_p^*} + 0{\cdot}1\left[B_h^* + (B_h^* + B_m^*)\frac{c_p^c - c_p^*}{c_p^*}\right](T_w - T_\infty)$$

for all cases except the case of hydrogen injected into air with free-stream Mach number of 12 [18], for which case reference temperatures are computed using

$$h^* \approx 0{\cdot}5\,(h_w + h_\infty) + 0{\cdot}2\,r_0^* \frac{u_\infty^2}{2} + 0{\cdot}1\left[B_h^* + (B_h^* - 2B_m^*)\frac{c_p^c - c_p^*}{c_p^*}\right](h_w - h_\infty).$$

The reference compositions are computed using

$$c^{a*} = \frac{M^a}{M^a - M^c}\,\frac{\ln M_\infty/M_w}{\ln c^a_\infty M_\infty/c^a_w M_w}$$

Table 3. Comparison of parameters

	Couette flow	Boundary-layer flow
Reference enthalpy, h^*	$\frac{1}{2}(h_w + h_\infty) + \frac{1}{6} r_0^* \frac{u_\infty^2}{2} + \frac{1}{12}\left[B_h^* + (B_h^* - 2B_m^*)\frac{c_p^c - c_p^*}{c_p^*}\right](h_w - h_\infty)$	$0{\cdot}5(h_w + h_\infty) + 0{\cdot}2\, r_0^* \frac{u_\infty^2}{2} + 0{\cdot}1\left[B_h^* + (B_h^* - 2B_m^*)\frac{c_p^c - c_p^*}{c_p^*}\right](h_w - h_\infty)$
Reference concentration, c^{a*}	$\frac{M^a}{M^a - M^c}\frac{\ln M_\infty/M_w}{\ln c^a_\infty M_\infty/c^a_w M_w}$	$\frac{M^a}{M^a - M^c}\frac{\ln M_\infty/M_w}{\ln c^a_\infty M_\infty/c^a_w M_w}$
Blowing rate, B_f^*	$\frac{(\rho v)_w}{\rho^* u_\infty} Re^*$	$\frac{(\rho v)_w}{\rho^* u_\infty}\frac{\sqrt{(Re^*)}}{0{\cdot}332}$
Blowing rate, B_m^*	$\frac{(\rho v)_w}{\rho^* u_\infty} Re^*\, Sc^*$	$\frac{(\rho v)_w}{\rho^* u_\infty}\frac{\sqrt{(Re^*)}}{0{\cdot}332} Sc^{*0\cdot64[C_f\ (Re^*)/0\cdot664]^{1/6}}$
Blowing rate, B_h^*	$\frac{(\rho v)_w}{\rho^* u_\infty} Re^*\, Pr^*$	$\frac{(\rho v)_u}{\rho^* u_\infty}\frac{\sqrt{(Re^*)}}{0{\cdot}332} Pr^{*0\cdot64[C_f^* \sqrt{(Re^*)}/0\cdot664]^{1/6}}$
Ordinate for momentum transfer	$\frac{C_f^*}{2} Re^*$	$\frac{C_f^*}{2}\frac{\sqrt{(Re^*)}}{0{\cdot}332}$
Ordinate for mass transfer	$C_m^*\, Re^*\, Sc^*$	$C_m^* \frac{\sqrt{(Re^*)}}{0{\cdot}332} Sc^{*0\cdot64[C_f^* \sqrt{(Re^*)}/0\cdot664]^{1/6}}$
Ordinate for energy transfer	$C_h^*\, Re^*\, Pr^*$	$C_h^* \frac{\sqrt{(Re^*)}}{0{\cdot}332} Pr^{*0\cdot64[C_f^* \sqrt{(Re\ )}/0\cdot664]^{1/6}}$
Ordinate for recovery factor	$\frac{r^*}{Pr^*}$	$\frac{r^*}{Pr^{*1\cdot2-0\cdot7[C_f^* \sqrt{(Re\ )}/0\cdot664]^{1/6}}}$
Abscissa for momentum transfer	B_f^*	B_f^*
Abscissa for mass transfer	B_m^*	B_m^*
Abscissa for energy transfer	$B_h^* + B_h^* \frac{c_p^c - c_p^*}{c_p^*}$	$B_h^* + 0{\cdot}6\, B_h^* \frac{c_p^c - c_p^*}{c_p^*}$
Abscissa for recovery factor	$B_h^* - B_f^* + (B_h^* + \frac{1}{2} B_m^*)\frac{c_p^c - c_p^*}{c_p^*}$	$B_h^* - B_f^* + (B_h^* + 0{\cdot}5 B_m)\frac{c_p^c - c_{p\infty}}{c_p^*}$

for all cases, even though the linear approximation

$$c^{c*} \approx 0{\cdot}5(c_w^c + c_\infty^c)$$

would be adequate for all coolants examined except H_2 and I_2. The blowing-rate parameters B_f^*, B_m^*, and B_h^*, all ordinates, and the abscissae for the momentum-transfer and mass-transfer correlations are changed only in accordance with the results of the correlation of constant-property flows. Better energy-transfer and recovery-factor correlations are obtained, however, if one replaces the abscissae

$$B_h^* + B_h^* \frac{c_p^c - c_p^*}{c_p^*}$$

and

$$B_h^* - B_f^* + (B_h^* + \tfrac{1}{2} B_m^*) \frac{c_p^c - c_p^*}{c_p^*}$$

respectively by

$$B_h^* + 0{\cdot}6\, B_h^* \frac{c_p^c - c_p^*}{c_p^*}$$

and

$$B_h^* - B_f^* + (B_h^* + \tfrac{1}{2} B_m^*) \frac{c_p^c - c_{p_\infty}}{c_p^*}.$$

These two abscissae changes are accepted as empirical results.

In a correlation of heat-transfer rates using the concept of a reference state, the value of the recovery temperature (or enthalpy) used in the Stanton number must be consistent with the reference state, i.e. the proper value of the recovery temperature is that value which would be obtained for a constant-property fluid with properties corresponding to the reference state. The recovery temperature obtained for variable-property fluids is to be used only in the Stanton number for $T_w = T_r$; the reference state for $T_w \neq T_r$ is not, in general, the same as the reference state for $T_w = T_r$. Hence, of the heat-transfer calculation results presented by Eckert *et al.* [18], only those results for $T_w = T_r$ are used here.†

Correlations obtained using the aforementioned reference states and parameters are presented as Figs. 12–15. Note that the curve drawn in Fig. 15 joins smoothly with the curve drawn in Fig. 7. Transfer rates and recovery factors for fluids with variable properties are given now as functions of blowing rates for several coolants and several speeds (cf. Table 4) by only two curves to an approximation adequate for many engineering applications.

USE OF REFERENCE-STATE CONCEPT IN OTHER FLOW MODELS

The usefulness of the concept of a reference state in correlations of transfer rates and recovery factors for laminar flows along flat plates with mass additions has been demonstrated. How useful would this concept be in attempts to correlate results for other flow models?

For laminar flows over cones with mass additions at the wall, it is believed that the concept of a reference state would be useful equally as for laminar flows over flat plates. For flows without mass additions at the wall, no difficulties were encountered in the application of the reference–temperature equation obtained for flows over flat plates to the case of flows over cones [22].

Also for laminar flows with pressure gradients and mass transfer, no difficulties are expected in the application of the reference–state equations developed here. For flows without mass transfer Romig [23] applied successfully the reference-enthalpy equation obtained for flows over flat plates to the case of stagnation-point heat transfer in hypersonic flow, whereas Eckert and Tewfik [24] used successfully the same equation in descriptions of heat-transfer distributions for hemisphere-cylinder bodies and flat-nosed plates in dissociated air. Gross *et al.* [13], examining

† Examining the results of Eckert *et al.* [18] for $T_w \neq T_r$, it was discovered that the curve for $T_w = 6T_\infty$ and $Ma_\infty = 12$ in Fig. 10 had been shifted inadvertantly toward the left-hand side of the figure. An examination of the calculation results presented in Table 2 of [21] revealed that the abscissa range for this curve should be (using nomenclature of [18]) $0{\cdot}1 \leq W_w \leq 0{\cdot}8$ rather than $0 \leq W_w \leq 0{\cdot}6$. (This error is retained apparently in Fig. 12 of [13].) Also, recalculating the Schmidt numbers presented in Fig. 1 of [18] and [21], it is concluded that the free-stream Schmidt number is approximately 0·197 rather than 0·221 as implied in the aforementioned figure. The courtesy of Dr. J. F. Gross, of the RAND Corporation, in lending his copy of [21] to the author after other attempts to obtain a copy had failed, is appreciated greatly.

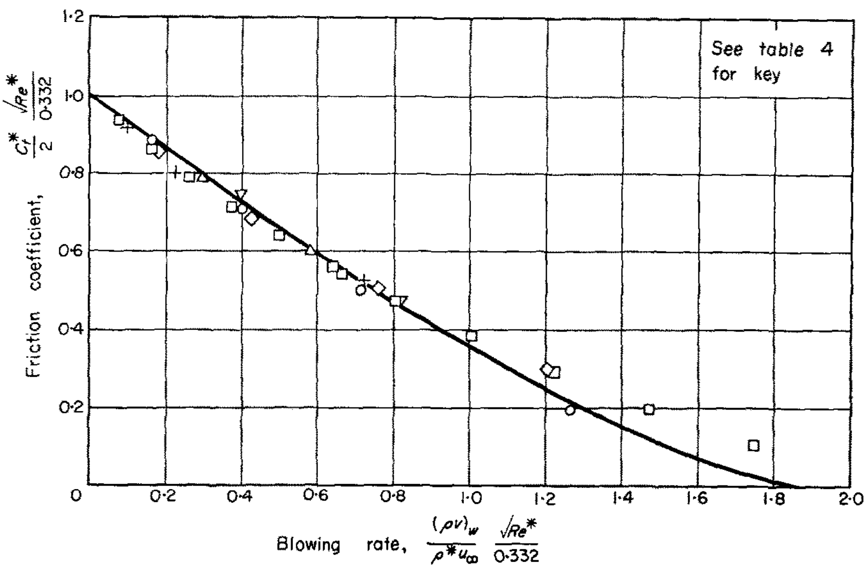

FIG. 12. Friction coefficient for fluids with variable properties as function of blowing rate and for several coolants: result of attempt to describe results for several coolants and several speeds by a single curve.

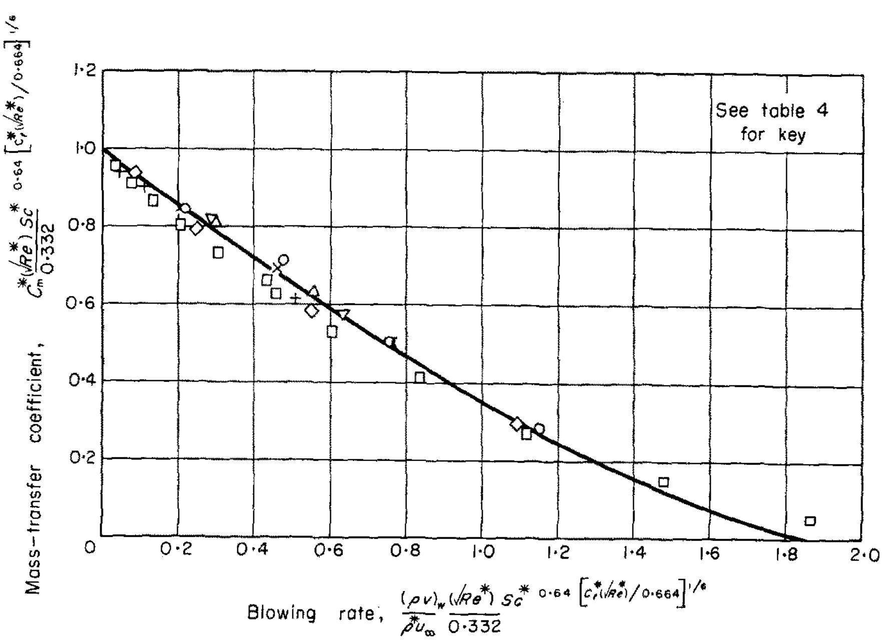

FIG. 13. Mass-transfer coefficient for fluids with variable properties as function of blowing rate and several coolants: result of attempt to describe results for several coolants and several speeds by a single curve.

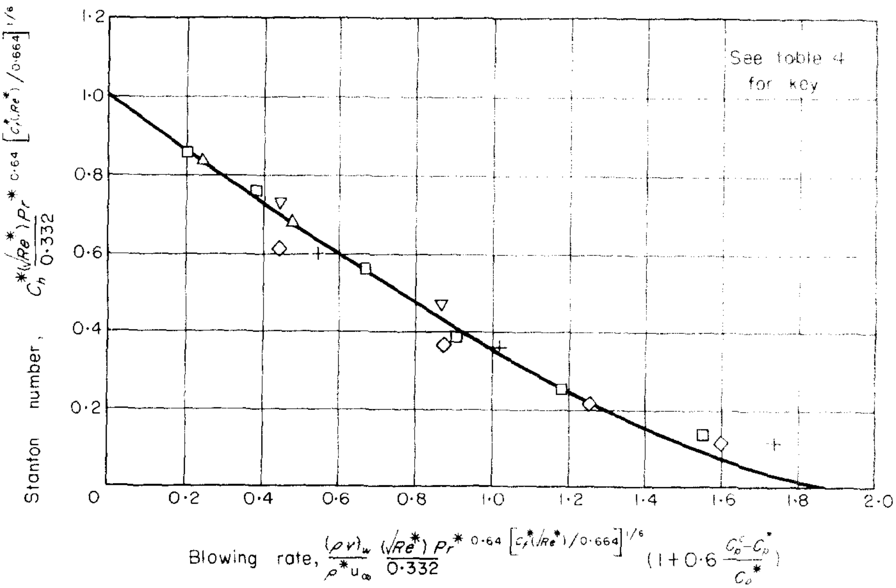

FIG. 14. Stanton number for fluids with variable properties as function of blowing rate and for several coolants: result of attempt to describe results for several coolants and several speeds by a single curve.

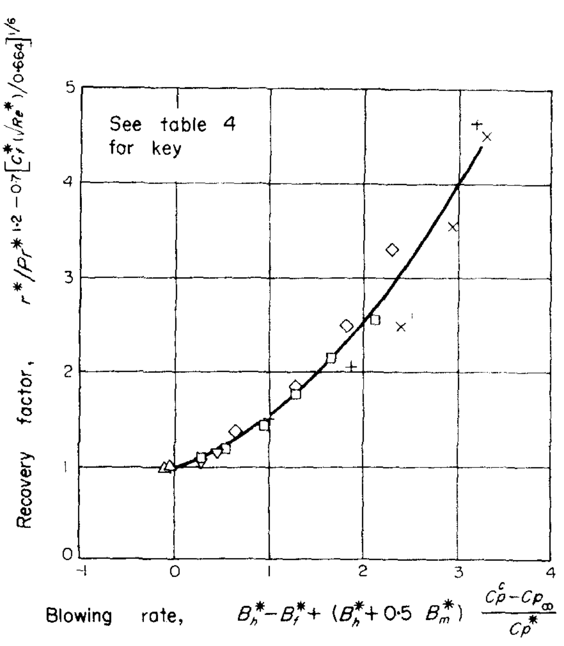

FIG. 15. Recovery factor for fluids with variable properties as function of blowing rate and for several coolants: result of attempt to describe results for several coolants and several speeds by a single curve.

calculations by Hayday [25] for plane stagnation flow with addition of a foreign gas at the wall, conclude that their empirical reference-temperature molecular-weight correlations, obtained from examinations of calculations for flows over flat plates, may be used for flows with pressure gradients to predict heat-transfer rates for fluids with variable properties from results for fluids with constant properties.

Eckert [10] found that, for flows without mass additions at the wall, the reference–temperature expression used for laminar flows correlated successfully also the results for turbulent flows. Hence, one might be encouraged to attempt to correlate also results for turbulent flows with mass transfers using the reference–state expressions developed here for laminar flows. However, since the present state of knowledge of turbulent flows with mass transfer is such that one would question the value of any reference–state expressions obtained examining analytical results, one is led to examine experimental results. A search of the available literature reveals that reliable data for turbulent flows are scarce; a correlation of the available useful data (heat-transfer rates for nitrogen injected into air) is to be presented in a separate communication.

Table 4. *Key to Figs.* 8–15

Symbol	Coolant	M^a/M^c	$c_p{}^c{}_\infty/c_p{}^a{}_\infty$	Pr_∞	Sc_∞	Le_∞	Ma_∞	Ref.	Author
×	H_2	14·5	13·8	0·73	0·197	0·303	12	18	Eckert, *et al.*
+	H_2	14·3	14·2	0·70	0·195	0·279	3	19	Sziklas
□	He	7·24	5·10	0·73	0·230	0·315	—	17	Baron
◇	He	7·25	5·18	0·70	0·211	0·302	3	19	Sziklas
△	CO_2	0·66	1·00	0·73	1·04	1·42	—	17	Baron
▽	H_2O	1·61	1·85	0·70	0·666	0·951	6	19	Sziklas
○	I_2	0·114	—	—	1·63	—	—	13, 20	Gross

Notes: 1. Results by Eckert *et al.* and Sziklas are for case in which free-stream temperature is 392°R and wall temperature is recovery temperature. Results by Baron and Gross are independent of temperature (and Mach number).

2. Main-stream gas is air.

3. Soret (thermal-diffusion) and Dufour (diffusion-thermo) effects are neglected. Baron [26] has shown that these effects are important in some cases. The author believes that, as soon as a sufficient number of either experimental results or exact calculations including these effects are available, an attempt to extend the use of reference states to include these cases would be worthwhile.

For the case in which chemical reactions occur, the fact that information concerning temperature and concentration distributions is essential to the use of the concept of a reference state complicates the application of this concept. This information is available readily only in simple cases, e.g. the case in which the chemical reaction rates are fast, so that the reaction rate is controlled by the mass-diffusion rates and the reaction occurs in a flame sheet [27]. The application of the concept of a reference state to this simple case has been examined; situations in which the chemical reactions occur chiefly near the wall or near the outer edge of the boundary-layer have been discussed briefly [28]; arbitrary locations of the flame sheet are to be discussed in a separate communication.

CONCLUSIONS

From examinations of the results of the analysis of laminar Couette flows with mass transfer and the results of exact calculations for laminar boundary-layer flows with mass transfer, the following conclusions are drawn:

(1) The reference composition appropriate for use in relations developed for fluids with constant properties must be calculated, in general, using

$$c^{a*} = \frac{M^a}{M^a - M^c}\frac{\ln M_\infty/M_w}{\ln c_\infty{}^a M_\infty/c_w{}^a M_w}.$$

The linearized expression

$$c^{c*} = \tfrac{1}{2}\,(c_w{}^c + c_\infty{}^c)$$

may be used only for low blowing rates or small molecular-weight differences.

(2) The reference temperature appropriate for use in relations developed for fluids with constant properties may be calculated using either

$$T^* \approx 0{\cdot}5\,(T_w + T_\infty) + 0{\cdot}2\,r_0{}^*\,\frac{u_\infty{}^2}{2c_p{}^*} + 0{\cdot}1\left[B_h{}^* + (B_h{}^* + B_m{}^*)\,\frac{c_p{}^c - c_p{}^*}{c_p{}^*}\right](T_w - T_\infty)$$

or

$$h^* \approx 0{\cdot}5\,(h_w + h_\infty) + 0{\cdot}2\,r_0{}^*\,\frac{u_\infty{}^2}{2} + 0{\cdot}1\left[B_h{}^* + (B_h{}^* - 2B_m{}^*)\,\frac{c_p{}^c - c_p{}^*}{c_p{}^*}\right](h_w - h_\infty)$$

the latter equation to be used for cases in which variations of the heat capacities with temperature are significant. These expressions are simple extensions of Eckert's expression for the reference enthalpy for a boundary-layer with no mass addition at the wall.

(3) The analogies between mass, momentum, and energy transfers are brought out clearly if one uses a mass-transfer coefficient proportional to the mass transferred by diffusion only, i.e. $(\rho v)_w(1 - c_w^c)$, rather than proportional to the total mass transferred, i.e. $(\rho v)_w$.

(4) The appropriate blowing parameters for use in correlations of the several transfer rates and the recovery factors for boundary-layer flows are indicated by results of the analysis of Couette flows. The blowing parameters for mass, momentum, and energy transfers include, as factors, the reciprocals of the mass-transfer coefficient, the friction coefficient and the Stanton number, all evaluated at zero blowing rate. The blowing parameter used in the correlation of recovery factors is a function of the several blowing parameters used in the correlations of the several transfer rates.

(5) The exponent 0·64 appearing in the Reynolds-analogy expressions

$$C_{h_0} = \frac{C_{f_0}}{2}\frac{1}{Pr^{0\cdot64}}$$

$$C_{m_0} = \frac{C_{f_0}}{2}\frac{1}{Sc^{0\cdot64}}$$

describes better available results of exact calculations than does the exponent 2/3 used frequently. For a Schmidt number of 0·2, $Sc^{2/3}$ differs from $Sc^{0\cdot64}$ by 4 per cent.

(6) For constant property fluids, for all blowing rates, and for Prandtl and Schmidt numbers from 0·5 to 1·5, the several relations between the transfer rates, the recovery factor, and the several blowing rates are described by only two curves to an approximation adequate for many engineering applications if one replaces the exponent 0·64 appearing in the Reynolds analogy expressions for zero blowing rate by $0\cdot64\,[c_f\sqrt{(Re)}/0\cdot664]^{1/6}$. This modification may be neglected either for small blowing rates or for Prandtl and Schmidt numbers close to unity.

(7) Using the reference states and parameters mentioned in conclusions (1–6) transfer rates and recovery factors for fluids with variable properties are given as functions of blowing rates for several coolants and several speeds by only two curves to an approximation adequate for many engineering applications. The curve for transfer rates is described to good approximation by the linear expression

$$y = 1 + \tfrac{2}{3}x$$

for abscissa values less than unity and by the quadratic expression

$$y = 1 + \tfrac{3}{4}x + \tfrac{1}{6}x^2$$

for all abscissa values. The curve for recovery factors is described to good approximation by

$$y = 1 + \tfrac{1}{3}x + \tfrac{2}{9}x^2$$

for the range of abscissa values examined.

(8) A Reynolds analogy for boundary-layer flows with mass additions at the wall does exist. It may be written:

If $Pr^* = Sc^* = 1$ and if $c_p^c = c_p^a$, then

$$\frac{C_f^*}{2} = C_m^* = C_h^* \quad \text{and} \quad r^* = 1.$$

Note that if $Pr^* = Sc^* = 1$ and if $c_p^c = c_p^a$, then the abscissas of Figs. 12–14 are equal and the abscissa of Figs. 15 may have only the value zero.

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Résumé—Deux cas simples d'écoulement laminaire avec transports simultanés de masse, de quantité de mouvement et d'énergie sont étudiés de façon à avoir un aperçu de la nature du phénomène de transport et des conditions thermodynamiques existant dans les couches limites à grande vitesse comportant la diffusion d'un gaz injecté à la paroi.

Dans l'écoulement de Couette laminaire, on peut faire des simplifications par suite de la nature des conditions aux limites et on peut obtenir des informations concernant les effets des différences de chaleurs spécifiques des deux espèces et les effets des variations, à partir de l'unité, des nombres de Prandtl, Schmidt et Lewis. Une expression exacte pour la composition de référence est obtenue dans le cas d'un écoulement isotherme; une expression linéaire de la temperature de référence est obtenue dans le cas où la viscosité est une fonction linéaire de la température et est indépendante de la composition. En première approximation, la composition de référence est la moyenne arithmétique des compositions aux limites tandis que l'expression de l'enthalpie de référence est une simple extension de l'expression d'Eckert de l'enthalpie de référence pour une couche limite laminaire lorsqu'il n'y a pas d'injection à la paroi. Dans l'écoulement laminaire à propriétés constantes, le fait que le fluide est à propriétés constantes apporte certaines simplifications et permet d'obtenir des renseignements sur les effets à caractère bidimensionnel de l'écoulement dans la couche limite. On peut alors combiner les renseignements obtenus dans ces deux cas simples et les appliquer au cas de l'écoulement de couche limite laminaire pour un fluide à propriétés variables. En basant toutes les propriétés du fluide sur l'état de référence suggéré dans l'étude de l'écoulement de Couette les coefficients de transport, de masse, d'énergie et de quantité de mouvement, et les facteurs thermiques pariétaux de fluides à propriétés variables, sont donnés en fonction des débits injectés de plusieurs refroidisseurs (y compris H_2, poids moléculaire 2, et I_2 poids moléculaire 254) et pour différentes vitesses (jusqu'à *Ma* = 12) uniquement par deux courbes, avec une précision suffisante pour les applications techniques. On a trouvé que l'analogie de Reynolds existait pour les écoulements de couche limite laminaire avec injection à la paroi.

Zusammenfassung—Zwei einfache Fälle der Laminarströmung bei gleichzeitigem Stoff-, Impuls- und Energietransport wurden untersucht, um einen Einblick zu erhalten in die Natur der Transportphänomene und thermodynamischen Bedingungen einer Hochgeschwindigkeitsgrenzschicht mit Diffusion eines an der Wand zugeführten Gases. Für die laminare Couetteströmung ergeben sich Vereinfachungen aus der Natur der Grenzbedingungen und man erhält Aussagen über den Einfluss unterschiedlicher Wärmekapazitäten beider Medien und der Abweichungen der Prandtl-, Schmidt- und Lewiszahl von Eins. Ein exakter Ausdruck für die Bezugszusammensetzung lässt sich für isotherme Strömung erhalten; ein linearisierter Ausdruck für die Bezugstemperatur ergibt sich, wenn die Viskosität eine lineaire Funktion der Temperatur und unabhängig von der Zusammensetzung ist. In erster Näherung gilt für den Ausblasestrom als Bezugszusammensetzung das arithmetische Mittel der Zusammensetzungen an den beiden Oberflächen, während die Bezugsenthalpie eine einfache Erweiterung des Eckertschen Ausdruckes für die Bezugsenthalpie der laminaren Grenzschicht ohne Stoffzugabe an der Wand darstellt. Für die laminare Grenzschichtströmung mit konstanten Eigenschaften sind Vereinfachungen infolge dieser Konstanz möglich und es ergeben sich Aufschlüsse über den Einfluss des zweidimensionalen Charakters der Grenzschichtströmung. Die aus den beiden einfachen Fällen erhaltenen Informationen wurden kombiniert und auf die laminare Grenzschichtströmung mit veränderlichen Flüssigkeitseigenschaften angewandt. Bezieht man alle Flüssigkeitseigenschaften auf die in der Untersuchung der Couetteströmung vorgeschlagenen Bezugsgrössen, ergeben sich Stoff-, Impuls- und Energietransportgeschwindigkeiten und der Rückgewinnfaktor für Flüssigkeiten veränderlicher Eigenschaften als Funktion der Transportwerte. Für verschiedene Kühlmittel (einschliesslich H_2, Molekulargewicht 2 und I_2, Molekulargewicht 254) und für verschiedene Geschwindigkeiten (bis *Ma* = 12) liessen sich die Abhängigkeiten in nur zwei Diagrammen angeben, die für viele ingenieurmässige Anwendungen angemessene Näherungen darstellen. Eine Reynoldsanalogie für Grenzschichtströmungen mit Stoffzugabe liess sich aufzeigen.

Аннотация—Расмотрены два простых случая ламинарного потока при одновременном переносе массы, момента и энергии, прежде всего, с целью выяснения сущности механизма явлений переноса и термодинамики высокоскоростного пограничного слоя при диффузии газа, подаваемого на стенку. Упрощения для ламинарного течения Куэтта рассматриваются как результат самой природы граничных условий. Приводятся данные о влиянии разности теплоемкостей двух образцов и отклонений от единицы чисел Шмидта, Прандтля и Льюиса. Получено точное выражение исходного состава для изотермического потока, а также линейное выражение для исходной температуры для случая, когда вязкость есть линейная функция температуры, независящая от состава. Для небольших скоростей вдува исходный состав принимается как среднеарифметическое составов на двух поверхностях, а выражение исходной энтальпии как простое обобщение формулы Эккерта для пограничного слоя при отсутствии подачи массы. Для упрощения задачи ламинарного пограничного слоя свойства жидкости принимаются постоянными. Получены данные для двухмерной задачи. Затем данные для этих двух простых случаев объединяются и применяются для ламинарного пограничного слоя жидкости с переменными свойствами. Применяя исходное положение течения Куэтта, для всех свойств жидкостей, скорости переноса массы, энергии и момента, а также явления восстановления даются как функции скоростей вдува нескольких охладителей (включая H_2 с молекулярным весом 2 и I_2 с молекулярным весом 254) для нескольких скоростей (до *Ma* = 12) в виде двух кривых. Такая аппроксимация удобна для многих инженерных приложений. Обнаружена аналогия Рейнольдса для пограничного слоя с подачей массы на стенке.